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LETTER TO THE EDITOR

Experimental observation of stochastic postponements of critical onsets in a bistable system

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Abstract. We have employed a novel type of analogue electronic simulator for an experimental study of a nonlinear stochastic differential equation representative of a large class of bistable systems. The results obtained are in good agreement with theoretical predictions derived from the Stratonovic version of the white noise Fokker–Planck equation. The work is believed to be of relevance to a wide range of topics in physics, chemistry, biology and other branches of science.

Nonlinear dynamical systems that exhibit a continuous instability can often be influenced to a quite dramatic extent by the external modulation of a control parameter. The modulation may sometimes be an unavoidable influence of the system's surroundings; in other cases, it is introduced deliberately by the experimenter; and its frequency content can lie anywhere between the two extremes of white noise on the one hand, or a single frequency sine wave on the other. Of particular interest is the fact that in many cases, covering a very wide range of scientific contexts, the effect of the modulation is to shift the onset of the transition or instability to a *larger* (average) value of the control parameter in question: that is, in the direction opposite to that suggested by intuition. Such effects have been found, or are to be anticipated, in such diverse areas as: quantised turbulence in superfluid ^{4}He (Oberly and Tough 1972, Moss and Welland 1982); classical hydrodynamics (Donnelly *et al* 1962, Ahlers *et al* 1984); electronics (Kabashima and Kawakubo 1979); optics (Bowden *et al* 1981); liquid crystals (Kai *et al* 1979, Brand and Schenzle 1980, Kawakubo *et al* 1981) where, in relation to the penultimate citation, it is important to note (San Miguel and Sancho 1981) that, in reality, the noisy multiplicative parameter is itself nonlinear; chemical equilibria (Roux *et al* 1983); and tumour immunology (Lefever and Horsthemke 1979).

Although these phenomena are not yet fully understood theoretically, it appears probable, in view of their seemingly universal nature, that (contrary to alternative explanations offered in some of the earlier papers) they are an inherent property of all systems displaying a certain class of nonlinearity. In this letter, we present a preliminary report of an experimental study of a model nonlinear system belonging to the class in question, for which a detailed theoretical analysis already exists, and we show that it does indeed exhibit the anticipated stochastic postponement effect.

One of the more fruitful theoretical approaches (but see also Graham and Schenzle 1982) to the white noise problem in general has entailed the use of the appropriate Fokker–Planck equation for calculation of the stationary probability density function whose maxima, representing 'most probable' values of the parameter in question, are then investigated as a function of the modulation intensity and other parameters of the system. This is the philosophy adopted by, particularly, Horsthemke, Lefever and their co-workers. A detailed exposition and review will be found in Horsthemke and Lefever (1984). Welland and Moss (1982) applied this type of approach to the cubic bistable system described by the stochastic differential equation

$$\mathrm{d}x/\mathrm{d}t = -x^3 + \lambda_t x^2 - Qx + R = f(x, t) \quad (1)$$

where

$$\lambda_t = \lambda + \sigma\xi_t \quad (2)$$

is the noisy control parameter of mean λ and variance σ^2, and Q and R are constants. This equation can be regarded as a general example of a system with a continuous instability. For $\sigma = 0$, with an appropriate choice of Q and R, there is a range of λ for which the equilibrium value of x is treble valued (two stable roots and one unstable one). When $\sigma \neq 0$, x is of course no longer deterministic; but it can still be described by its statistical density $\rho(x)$. Welland and Moss (1982) showed that, in the presence of Gaussian white noise, the stationary density should be given by

$$\rho_s(x) = \frac{N}{x^{2(\nu+1/\sigma^2)}} \exp\left[\frac{2}{\sigma^2}\left(-\frac{\lambda}{x} + \frac{Q}{2x^2} - \frac{R}{3x^3}\right)\right] \quad (3)$$

where N is a normalising constant and $\nu = 1$ or 2, depending on whether one chooses to use the Stratonovic or Ito version respectively of the white noise Fokker–Planck equation. By analogy with recent experiments on the so-called genetic model (Smythe *et al* 1983a), we can anticipate that the appropriate choice in the present case will probably also turn out to be $\nu = 1$. The most interesting point about (3) is that it predicts a shift with σ of the positions x_m of the extrema of $\rho_s(x)$, and it is straightforward to demonstrate that these are given by the roots of

$$(1 + \nu\sigma^2)x_m^3 - \lambda x_m^2 + Qx_m - R = 0. \quad (4)$$

Furthermore, an increase of σ tends to stabilise the system in its lower state, corresponding to a postponement of the transition; and, for a sufficiently large value of σ the transition is completely annihilated. This type of behaviour is, of course, highly reminiscent of that observed in some of the experiments quoted above.

A quantitative comparison of these ideas with experiment is difficult to achieve, however, first because it is in general not known how accurately (1) (or an equation of similar type) can be expected to model any given natural phenomenon and, secondly, because it is usually not possible to measure the density function itself wherein resides most of the detailed information about the stochastic processes under study. Usually, it is mean values such as $\langle x \rangle$ or $\langle x^2 \rangle$ that are measured in experiments, and these inevitably introduce ambiguities of interpretation. For these reasons, we have elected to study the effect of noise on an analogue simulator that is designed specifically to follow (1) as accurately as possible, under conditions such that $\rho_s(x)$ and $x_m(\sigma, \lambda)$ can be measured explicitly for comparison with (3) and (4).

A block diagram of the simulator, for the case of $Q = 3$, $R = 0.7$ in (1), is shown in figure 1. It is similar in general design to that employed earlier to search for a NIPT in the genetic model (Smythe *et al* 1983b), has an overall scale factor of unity and is constructed entirely from standard commercially available analogue components. Gaussian white noise from the noise generator is bandwidth-limited by means of a single-pole filter of time constant τ_N. It is straightforward to demonstrate (Moss *et al*

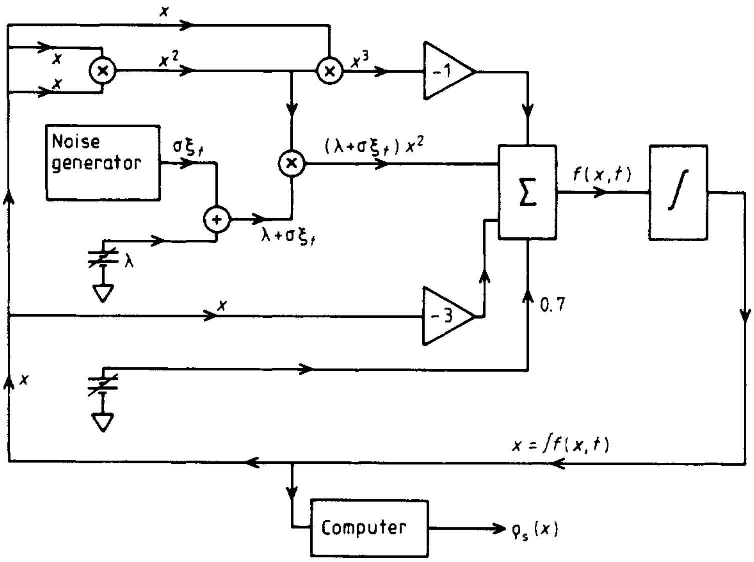

Figure 1. Block diagram of the electronic analogue simulator for equation (1).

1984) that, under these 'real white noise' conditions, the σ^2 that appears in (3) and (4) must be replaced by $(2\tau_N/\tau_I)V_N^2$ where V_N is the RMS amplitude of the noise applied to the simulator and $\tau_I \gg \tau_N$ is the time constant of the integrator. In practice, we have set $\tau_I = 1.2$ ms, $\tau_N = 12$ μs for most of the measurements. The noise is therefore perceived by the simulator as being white.

The DC ($\sigma = 0$) response of the simulator, modelling the deterministic version of (1), is shown by the circled points in figure 2. The quality of agreement with the

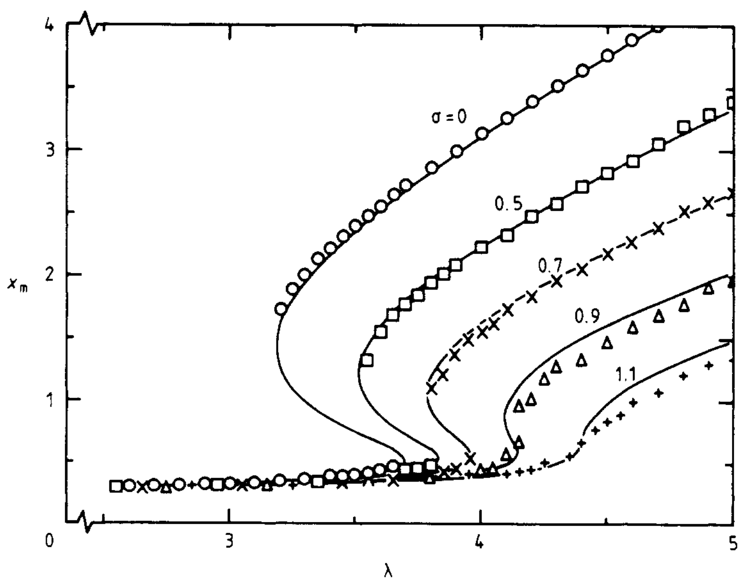

Figure 2. Plots of the maxima x_m in the stationary density function measured for the electronic simulator shown in figure 1. The measured values are plotted against λ for various values of σ. The full curves are plots of equation (4).

associated full curve, representing (1), may be considered quite reasonable in view of the known small non-idealities associated with the individual components (particularly the analogue multipliers) that comprise the circuit.

With noise applied to the simulator, x is of course a fluctuating quantity, and we then measure its density function $\rho(x)$ by means of the Nicolet 1080 computer in the manner described previously (Smythe *et al* 1983b). Some selected experimental densities are plotted for $\lambda = 3.6$ in figure 3. As σ is increased from zero, $\rho(x)$ starts to

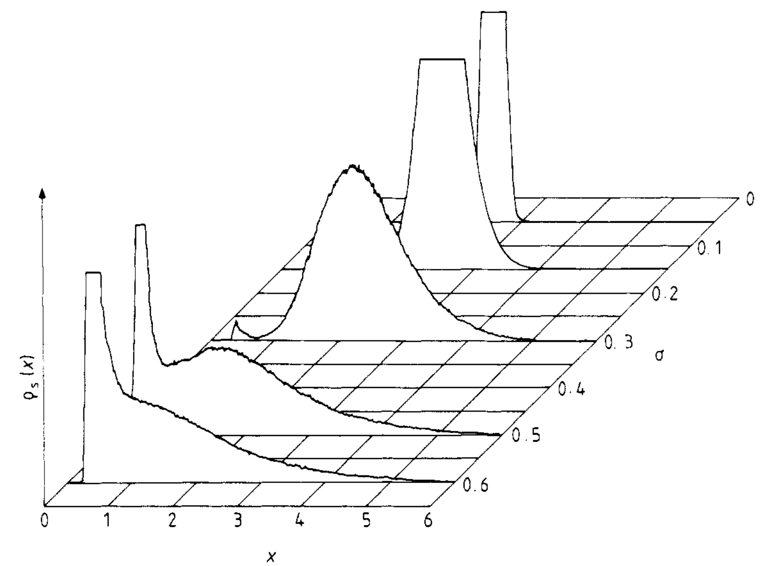

Figure 3. Some selected examples of stationary density functions $\rho_s(x)$ measured with $\lambda = 3.6$ for the electronic simulator shown in figure 1. As the variance σ^2 of the applied noise is increased the density broadens, shifts, becomes bimodal and finally, for very large σ, converges towards a delta function near to the lower root of the deterministic ($\sigma = 0$) version of equation (1). The shapes of these densities are found to be in good agreement with the Stratonovic ($\nu = 1$) version of equation (3).

spread from its initial delta function, but at first remains centred on the deterministic solution. With a further increase of σ, x_m decreases and a second maximum eventually starts to develop at a value of x that is close to the lower root of the deterministic equation. For sufficiently large σ, the upper maximum is completely overwhelmed, and the density tends towards a delta function centred on the lower root.

Similar behaviour occurs even when λ is *beyond* the position of the upward transition in the deterministic solution, corresponding to the predicted stochastic postponement effect. We have measured $x_m(\sigma, \lambda)$ by fitting parabolae by the method of least squares to the regions close to the maxima of density functions such as those of figure 3 for a wide range of σ and λ, with the results shown by the $\sigma \neq 0$ points of figure 1. The associated full curve in each case represents the Stratonovic version of (4): that is, with $\nu = 1$. It should be emphasised that there are no adjustable parameters in these comparisons. The discrepancies at large λ and σ quite probably arise from 'clipping' of the largest voltage excursions in the section of the circuit that deals with the quadratic term; but, apart from this, the level of agreement may be regarded as remarkably good. A careful comparison of $\rho_s(x)$ with (3) for $\nu = 1$ and $\nu = 2$ quite clearly favours the ($\nu = 1$) Stratonovic stochastic calculus, consistent with our earlier conclusion (Smythe *et al* 1983a) that, even though the Ito approach is correct from a strictly mathematical point of view, it is the Stratonovic interpretation of the white noise process that describes what actually happens in nature: this aspect of the work will be discussed elsewhere (McClintock and Moss 1984) in relation to detailed comparisons of the measured and theoretical densities.

When λ is being changed at constant σ we have found that, for small σ, the transition remains hysteretic on the time scale of the experiment. For larger σ, although the system always remains centred on either the upper or the lower state as before, it may switch spontaneously between them during the period τ_A (typically ten minutes) necessary for acquisition of a density function. Reproducible measurements of bimodal densities, such as those for $\sigma = 0.3$, 0.5 in figure 3, become possible only when σ is large enough for the mean first passage times τ_p between the states to have become very much shorter than τ_A. We suspect that the magnitude of τ_p is closely related *inter alia* to the amplitudes of the low frequency modulation components so that, if these were attenuated, as compared to the white noise of the present studies, the transition would appear to retain its hysteresis despite being shifted to the larger values of λ shown in figure 2.

We believe that these results represent significant progress towards an understanding of the remarkably catholic stochastic postponement effect referred to at the beginning of this letter. Much work still remains to be done, however. Experimentally, even in relation to the present simulator, it will be necessary, as mentioned above, to study the effect of coloured noise of various kinds and, in particular, to investigate time-dependent effects. Theoretically, it remains highly desirable to achieve a more satisfactory understanding of the relationship (Graham and Schenzle 1982) between bifurcation theory and the Fokker-Planck approach from which (2)-(4) were derived. We can, however, regard the excellent agreement between experiment and theory in figure 2 as constituting a convincing verification of the calculations of Welland and Moss (1982) thereby, in turn, confirming the hypothesis that the stochastic postponement of transitions is a universal phenomenon inherent in the nonlinearities of the class of bistable systems under discussion.

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